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Answer to an open problem proposed by R Metzler and J Klafter

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Abstract
In a positive answer to the open problem proposed by R Metzler and J Klafter, it is proved that the asymptotic shape of the solution for a wide class of fractional Fokker–Planck equations is a stretched Gaussian for the initial condition being a pulse function in the homogeneous and heterogeneous fractal structures, whose mean square displacement behaves like $\langle(\Delta x)^2(t)\rangle \sim t^\gamma$ and $\langle(\Delta x)^2(t)\rangle \sim x^{-\theta}t^\gamma (0 < \gamma < 1, -\infty < \theta < +\infty)$, respectively.

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1. Introduction

Anomalous diffusion in one dimension is characterized by the occurrence of a mean square displacement of the form

$$X_0^2 = \langle(\Delta x)^2(t)\rangle_0 = \frac{2K_\gamma}{\Gamma(1+\gamma)}t^\gamma. \tag{1}$$

The subscript zero in $\langle\cdot\rangle_0$ denotes the case when no external driving force is applied to the particle. When $\gamma \neq 1$ the diffusion is anomalous; the case $0 < \gamma < 1$ is called slow diffusion or sub-diffusion and $\gamma > 1$ is called super-diffusion.

Recently, in the literature, in order to describe diffusion processes in disordered media, some authors proposed an extension of the Fokker–Planck equation, which is called the fractional Fokker–Planck equation (FFPE) [1–10]. In [1], a fractional Fokker–Planck equation involving external field

$$\frac{\partial W(x,t)}{\partial t} = {}_0D_t^{1-\gamma} L_{\mathrm{FP}} W, \qquad 0 < \gamma < 1, \tag{2}$$

was presented to describe the anomalous transport close to thermal equilibrium, where $W(x, t)$ is the probability density function at position x at time t, and the Fokker–Planck operator

$$L_{\mathrm{FP}} = \frac{\partial}{\partial x}\left(\frac{V'(x)}{m\eta_\gamma} + K_\gamma \frac{\partial}{\partial x}\right) \tag{3}$$

with the external potential $V(x)$ [11] contains the anomalous diffusion constant K_γ and the anomalous friction coefficient η_γ with the dimension $[\eta_\gamma] = \sec^{\gamma-2}$; herein m denotes the mass of the diffusion particle, and

$${}_0D_t^{1-\gamma} W = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \mathrm{d}\tau \frac{W(x,\tau)}{(t-\tau)^{1-\gamma}}. \tag{4}$$

The fractional Fokker–Planck equation is shown [5] to be closely related to the Scher–Lax–Montroll (SLM) model [12–15] defined within the context of the continuous time random walk and to the collision models [15–18].

In [9], it is discussed that anomalous diffusion in a heterogeneous fractal medium in one dimension is characterized by the occurrence of a mean square displacement of the form

$$X_\theta^2 = \langle(\Delta x)^2(t)\rangle_0 \sim x^{-\theta} t^\gamma, \qquad 0 < \gamma < 1, \quad \theta = d_w - 2, \tag{5}$$

where $d_w > 2$ is the anomalous diffusion exponent [19]. By simple scaling consideration, equation (5) is equivalent to [20, 21]

$$X_\theta^2 = \langle(\Delta x)^2(t)\rangle_0 = \frac{2K_\gamma^\theta}{\Gamma(1+\gamma_\theta)} t^{\gamma_\theta}, \tag{6}$$

where

$$\gamma_\theta = 2\gamma/(2+\theta), \qquad 0 < \gamma_\theta < 1. \tag{7}$$

In equation (6) the anomalous diffusion coefficient K_γ^θ is introduced, which has the dimension $\left[K_\gamma^\theta\right] = \mathrm{cm}^2 \sec^{-2\gamma/(2+\theta)}$. It is easy to see that equation (6) reduces to equation (1) and K_γ^θ to K_γ when $\theta \to 0$. Indeed, θ can be any real number in this paper.

In [22], it is pointed out that the detailed structure of the propagator $W(x, t)$ (i.e. the probability density function) for the initial condition $\lim_{t\to 0+} W(x, t) = \delta(x)$ depends generally on the special shape of the underlying geometry. However, the interesting part of the propagator has the asymptotic behaviour $\log W(x, t) \sim -c\xi^u$ where $\xi \equiv x/t^{\alpha/2} \gg 1$, which is expected to be universal. Here, $u = 1/(1-\alpha/2)$ with the anomalous diffusion exponent α which is the order of the fractional derivative. Does the stretched Gaussian universality hold? It is an open problem.

It would be interesting to clarify this issue. In section 4, in a positive answer to the open problem proposed by R Metzler and J Klafter [22], it is proved that the propagator has the asymptotic behaviour $\log W(x, t) \sim -c\xi^u$ for the solution of a wide class of the fractional Fokker–Planck equations (FFPE) defined below, where $\xi \equiv x/t^{\alpha/2} \gg 1$ and $u = 1/(1-\alpha/2)$.

2. FFPE with variable coefficient

In this section, we will derive the fractional Fokker–Planck equation with a variable coefficient involving the external potential $V(x)$ by using heuristic argument of Giona and Roman [23].

The relationship between the total flux of probability current $S(x, t)$ from time $t = 0$ to time t and the average probability density $W(x, t)$, considered as the input and output of the fractal system [24], should satisfy the following equation (cf [23]):

$$\int_0^t S(x,\tau)\,\mathrm{d}\tau = x^{d_f-1}\int_0^t K(t,\tau)W(x,\tau)\,\mathrm{d}\tau, \tag{8}$$

where d_f is the fractal dimension of the fractal structure of the system considered. This is a conservation equation containing an explicit reference to the history of the diffusion process on the fractal structure. Since we are dealing with stationary processes, we expect that $K(t,\tau)$ be a function of difference $t-\tau$ only, i.e. $K(t,\tau)=K(t-\tau)$. We say $K(t,\tau)$ the diffusion kernel. We assume that diffusion sets are underlying fractals (underlying fractals denote self-similar sets in [25] or net fractals in [26, 27]), and the diffusion kernel on the underlying fractal should behave as

$$K(t-\tau)=\frac{A_\alpha}{(t-\tau)^\alpha},\qquad 0<\alpha<1, \tag{9}$$

where α is a diffusion exponent and A_α is a constant that can be determined [27].

On the other hand, we propose that the probability current $S(x,t)$ on the above fractional structure satisfies the following structure equation:

$$S(x,t)=Bx^{d_f-1}x^{-\theta'}L_{\mathrm{FP}}^{K,\nu,\mu}W(x,t), \tag{10}$$

where $\nu,\mu\in\mathbb{R}$, $B>0$ is to be determined, θ' is a non-negative parameter, and

$$L_{\mathrm{FP}}^{K,\nu,\mu}W(x,t)=\frac{\partial}{\partial x}\left(\frac{V'(x)}{m\eta_\gamma}W(x,t)+Kx^{-\nu}\frac{\partial}{\partial x}x^{-\mu}W(x,t)\right), \tag{11}$$

where $K=K_\gamma^\theta$. From equations (8)–(10), we have

$${}_0D_t^\alpha W(x,t)=Gx^{-\theta'}L_{\mathrm{FP}}^{K,\nu,\mu}W(x,t), \tag{12}$$

where $G=B/\Gamma(1-\alpha)A_\alpha>0$. We call equation (12) a conjugate fractional Fokker–Planck equation with a variable coefficient.

By the theorem of the conjugate operator of the Riemann–Liouville fractional integrals and derivatives ([28], p 45), it follows from equation (12) that

$$W(x,t)=\frac{1}{\Gamma(\alpha)}\int_0^t Gx^{-\theta'}\frac{L_{\mathrm{FP}}^{K,\nu,\mu}W(x,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau, \tag{13}$$

for $W(x,t)\in I_0^\alpha(L_1)$. Hence, we obtain

$$\frac{\partial W(x,t)}{\partial t}=Gx^{-\theta'}{}_0D_t^{1-\alpha}L_{\mathrm{FP}}^{K,\nu,\mu}W(x,t). \tag{14}$$

This is a fractional Fokker–Planck equation (FFPE). Especially, when $G=1,\theta'=\nu=\mu=0$ and $\alpha=\gamma$, we have

$$\frac{\partial W(x,t)}{\partial t}={}_0D_t^{1-\gamma}L_{\mathrm{FP}}W. \tag{15}$$

This is just the fractional Fokker–Planck equation (2). Its derivation is different from that in [1, 29].

Furthermore, we suppose the probability density function $W(x,t)$ satisfying the following normalization condition:

$$\int_0^\infty \mathrm{d}x\cdot x^{d_f-1}W(x,t)=1, \tag{16}$$

and boundary condition

$$\lim_{x\to\infty}W(x,t)=0. \tag{17}$$

The extraction of moments $\langle(\Delta X)^n\rangle$ is defined by

$$\langle(\Delta X)^n\rangle=\int_0^\infty \mathrm{d}x\cdot x^{d_f-1}x^nW(x,t). \tag{18}$$

We can anticipate the relation between exponents $\alpha, \gamma, \theta, \nu, \mu$ and θ' by simple scaling considerations. From (14) we see that $t^\alpha \sim x^{2+\theta'+\nu+\mu}$, and according to (5) we require that $t^\gamma \sim x^{2+\theta}$, Thus,

$$\alpha = \frac{\gamma(2+\nu+\mu+\theta')}{2+\theta}, \qquad 0 < \frac{\gamma(2+\nu+\mu+\theta')}{2+\theta} < 1. \tag{19}$$

3. Fokker–Planck equations in frequency domain

In this section, we respectively give the solutions of the fractional Fokker–Planck equation (14) in the case of constant potentials, linear potentials, harmonic potentials, analytic potentials, logarithm potentials, pole potentials and generic potentials, which lead to force-free, uniform force, the Hookean force directed at the original, nonlinear force, hyperbolic force and mixed force, respectively. Using the Laplace transform of the fractional derivative, it follows from equation (14) that

$$\begin{aligned} x^2\frac{\partial^2 W(x,s)}{\partial x^2} &+ \left[x^{\nu+\mu}\frac{xV'(x)}{Km\eta_\gamma} - (\nu+2\mu)\right]x\frac{\partial W(x,s)}{\partial x} \\ &+ \left[\mu(\nu+\mu+1) + \frac{x^{\nu+\mu+2}V''(x)}{Km\eta_\gamma} - \frac{x^{\nu+\mu+\theta'+2}s^\alpha}{KG}\right]W(x,s) \\ &= -\frac{x^{\nu+\mu+\theta'+2}s^{\alpha-1}}{GK}W(x,0). \end{aligned} \tag{20}$$

We assume that the external potential at the original has the form

$$V(x) = \sum_{j=1}^{k} b_j x^{-j} + b\ln(x) + \sum_{n=0}^{\infty} a_n x^n, \tag{21}$$

where b_j, b, a_n can be zero.

In order to solve equation (20), it is convenient to perform the transform

$$y = A(s)x^v, \qquad W(x,s) = y^\delta Z(y) \tag{22}$$

to cast equation (20) into the second-order Bessel equation as

$$y^2\frac{\mathrm{d}^2 z}{\mathrm{d}y^2} + y\frac{\mathrm{d}z}{\mathrm{d}y} - (\lambda^2 + y^2)z(y) = -\frac{y^{2-\delta}}{s}W(x,0), \tag{23}$$

with parameter λ^2 under the following conditions:

$$v = \frac{1}{2}(\nu+\mu+\theta'+2), \qquad A(s) = \frac{1}{v}\left(\frac{s^\alpha}{KG}\right)^{1/2}, \tag{24}$$

$$\delta = \frac{1}{2v}\left\{\nu+2\mu+1 - \frac{1}{Km\eta_\gamma}\left(\frac{y}{A}\right)^{(\nu+\mu)/v}\left[-\sum_{j=1}^{k} jb_j\left(\frac{y}{A}\right)^{-j/v} + b + \sum_{n=1}^{\infty} na_n\left(\frac{y}{A}\right)^{n/v}\right]\right\}, \tag{25}$$

$$\begin{aligned} \lambda^2 = {} & \frac{\delta(1-v\delta)}{v} + \frac{\delta(\nu+2\mu)}{v} - \frac{\mu(\nu+\mu+1)}{v^2} \\ & - \frac{1}{v^2 Km\eta_\gamma}\left(\frac{y}{A}\right)^{(\nu+\mu-k)/v}\sum_{j=1}^{k} j(j+1-\delta v)b_j\left(\frac{y}{A}\right)^{(k-j)/v} \\ & + \frac{1}{v^2 Km\eta_\gamma}\left(\frac{y}{A}\right)^{(\nu+\mu)/v}\left[(1-\delta v)b - \sum_{n=1}^{\infty} n(n-1+\delta v)a_n\left(\frac{y}{A}\right)^{n/v}\right]. \end{aligned} \tag{26}$$

When $\mathrm{Re}(s^{\alpha/2})/y \gg 1$, i.e. $y/|A| \ll 1$, we have the asymptotic representations of δ, λ corresponding to different potentials as follows.

Case 1. $b_k \neq 0$, i.e. the origin is a pole of order k. It follows from (25) and (26) that if $\nu + \mu > k$,

$$\delta \approx \delta_0 = \frac{\nu + 2\mu + 1}{2v}, \qquad \lambda^2 \approx \lambda_0^2 = \frac{(1+\nu)^2}{4v^2} > 0, \tag{27}$$

which are independent of the external potentials (21) and if $\nu + \mu = k$,

$$\delta \approx \delta_0 = \frac{1}{2v}(k + \mu + 1 + \hat{b}_k), \qquad \hat{b}_k = \frac{kb_k}{Km\eta_\gamma}, \tag{28}$$

$$\lambda^2 \approx \lambda_0^2 = \frac{1}{4v^2}(k + 1 - \mu - \hat{b}_k)^2 = \frac{1}{4v^2}(1 + \nu - \hat{b}_k)^2 > 0, \tag{29}$$

which depend only upon the coefficient of the pole of order k.

Case 2. $b_j = 0\,(j = 1, 2, \ldots, k)$ but $b \neq 0$, i.e.

$$V(x) = b\ln(x) + \sum_{n=0}^{\infty} a_n x^n. \tag{30}$$

It follows from (25) and (26) that if $\nu + \mu > 0$,

$$\delta \approx \delta_0 = \frac{\nu + 2\mu + 1}{2v}, \qquad \lambda^2 \approx \lambda_0^2 = \frac{(1+\nu)^2}{4v^2} > 0, \tag{31}$$

which are independent of the external potentials (30) and if $\nu + \mu = 0$,

$$\delta \approx \delta_0 = \frac{1}{2v}(\mu + 1 - \hat{b}), \qquad \hat{b} = \frac{b}{Km\eta_\gamma}, \tag{32}$$

$$\lambda^2 \approx \lambda_0^2 = \frac{1}{4v^2}(1 - \mu + \hat{b})^2 > 0, \tag{33}$$

which depend only upon b, i.e. the coefficient of the term of the logarithmic singularity at the origin of the external potentials (30).

Case 3. $b_j = b = a_n = 0$ for $1 \leqslant j \leqslant k$ and $n < N$, but $a_N \neq 0$, i.e.

$$V(x) = \sum_{n=N}^{\infty} a_n x^n. \tag{34}$$

It follows from (25) and (26) that if $\nu + \mu + N > 0$,

$$\delta \approx \delta_0 = \frac{1}{2v}(\nu + 2\mu + 1), \qquad \lambda^2 \approx \lambda_0^2 = \frac{(1+\nu)^2}{4v^2} > 0, \tag{35}$$

which are independent of the external potentials (34) and if $\nu + \mu + N = 0$, i.e. $\nu + \mu = -N$,

$$\delta \approx \delta_0 = \frac{1}{2v}(1 + \nu - N - A_N), \qquad A_N = \frac{Na_N}{Km\eta_\gamma}, \tag{36}$$

$$\lambda^2 \approx \lambda_0^2 = \frac{1}{4v^2}(\mu + N - 1 - A_N)^2 = \frac{1}{4v^2}(1 + \nu + A_N)^2 > 0, \tag{37}$$

which depend only upon a_N, the coefficient of the first nonzero term of the external potentials (34).

Thus, equation (22) can be replaced by

$$y^2\frac{\mathrm{d}^2 z}{\mathrm{d}y^2} + y\frac{\mathrm{d}z}{\mathrm{d}y} - \left(\lambda_0^2 + y^2\right)z(y) = -\frac{y^{2-\delta_0}}{s}W(x,0). \tag{38}$$

4. Solutions and its properties of FFPE

In the case of $W(x,0)=\delta(x)$, equations (20) and (38) become

$$x^2\frac{\partial^2 W(x,s)}{\partial x^2}+\left[x^{\nu+\mu}\frac{xV'(x)}{Km\eta_\gamma}-(\nu+2\mu)\right]x\frac{\partial W(x,s)}{\partial x}$$
$$+\left[\mu(\nu+\mu+1)+\frac{x^{\nu+\mu+2}V''(x)}{Km\eta_\gamma}-\frac{x^{\nu+\mu+\theta'+2}s^\alpha}{KG}\right]W(x,s)$$
$$=-\frac{x^{\nu+\mu+\theta'+2}s^{\alpha-1}}{GK}\delta(x) \tag{39}$$

and

$$y^2\frac{\mathrm{d}^2 z}{\mathrm{d}y^2}+y\frac{\mathrm{d}z}{\mathrm{d}y}-\left(\lambda_0^2+y^2\right)z(y)=-\frac{y^{2-\delta_0}}{s}\delta\left(\left(\frac{y}{A(s)}\right)^{1/v}\right), \tag{40}$$

respectively.

The Delta function $\delta(x)$ can be approximated by the function

$$\delta_h(x)=\begin{cases}0, & x<0 \text{ or } x>h,\\ \dfrac{1}{h}, & 0<x\leqslant h.\end{cases} \tag{41}$$

Thus, equations (39) and (40) can also be approximated by

$$x^2\frac{\partial^2 W(x,s)}{\partial x^2}+\left[x^{\nu+\mu}\frac{xV'(x)}{Km\eta_\gamma}-(\nu+2\mu)\right]x\frac{\partial W(x,s)}{\partial x}$$
$$+\left[\mu(\nu+\mu+1)+\frac{x^{\nu+\mu+2}V''(x)}{Km\eta_\gamma}-\frac{x^{\nu+\mu+\theta'+2}s^\alpha}{KG}\right]W(x,s)$$
$$=-\frac{x^{\nu+\mu+\theta'+2}s^{\alpha-1}}{GK}\delta_h(x) \tag{42}$$

and

$$y^2\frac{\mathrm{d}^2 z}{\mathrm{d}y^2}+y\frac{\mathrm{d}z}{\mathrm{d}y}-\left(\lambda_0^2+y^2\right)z(y)=-\frac{y^{2-\delta_0}}{s}\delta_h\left(\left(\frac{y}{A(s)}\right)^{1/v}\right), \tag{43}$$

where δ_0 and λ_0^2 are respectively defined by (27)–(29); (31)–(33) and (35)–(37).

Using the operator method, we know that

$$Z_h^*(y)=\mathrm{e}^{-\int A_2\mathrm{d}y}\int \mathrm{e}^{\int(A_2-A_1)\mathrm{d}y}\left(\int f_h(y)\,\mathrm{e}^{\int A_1\mathrm{d}y}\,\mathrm{d}y\right)\mathrm{d}y \tag{44}$$

is a special solution of equation (43), where

$$A_1=-\frac{1}{2}\left[\frac{1}{y}-\sqrt{\frac{1}{y^2}+4\left(1+\frac{\lambda_0^2}{y^2}\right)}\right], \tag{45}$$

$$A_2=-\frac{1}{2}\left[\frac{1}{y}+\sqrt{\frac{1}{y^2}+4\left(1+\frac{\lambda_0^2}{y^2}\right)}\right], \tag{46}$$

$$f_h(y)=-\frac{y^{-\delta_0}}{s}\delta_h\left(\left(\frac{y}{A(s)}\right)^{1/v}\right). \tag{47}$$

To fit the boundary condition $\lim_{x\to+\infty} W(x,t) = 0$, i.e. $W(x,s) = 0(x \to +\infty)$, we should take an appropriate integral constant in equation (44) such that $Z_h^*(y) = 0$ as $y \to +\infty$. Thus, by the summability, we get the solution of equation (43),

$$Z_h(y) = C_h(s)K_{\lambda_0}(y) + Z_h^*(y), \tag{48}$$

where

$$K_{\lambda_0}(y) = \mathrm{e}^{-y}\left(\frac{\pi}{2y}\right)^{1/2}\left[1 + O\left(\frac{1}{y}\right)\right] \tag{49}$$

in the domain $|\arg y| < \frac{3}{2}\pi$ is the modified Bessel function of second order, and $C_h(s)$ is to be determined.

So the solution of equation (42) is given by

$$W_h(x,s) = C_h(s)y^{\delta_0}K_{\lambda_0}(y) + y^{\delta_0}Z_h^*(y), \qquad y = A(s)x^v. \tag{50}$$

By the normalization condition of $W(x,t)$, i.e. $\int_0^\infty \mathrm{d}x \cdot x^{d_f-1}W(x,s) = \frac{1}{s}$, it is not difficult to see that

$$C_h(s) = G's^{(\alpha d_f/2v)-1} + G''s^{\alpha(d_f/2v)+\alpha)}h^{(d_f-1+2v)}[1 + O(h^{2v})]. \tag{51}$$

Thus, if $d_f - 1 - 2v > 0$, i.e. $\nu + \mu > -(d_f + 1 - \theta')$, noting that $\lim_{h\to 0} C_h(s) = C(s)$, where

$$C(s) = G's^{(\alpha d_f/2v)-1}, \qquad G' = v^{1-d_f/v}\big/\big[C_{\lambda_0}\big(GK_\gamma^\theta\big)^{d_f/2v}\big], \tag{52}$$

$C_{\lambda_0} = \int_0^\infty \mathrm{d}y \cdot y^{\frac{d_f}{v}+\delta_0-1}K_{\lambda_0}(y)$ is a constant since [30]

$$\int_0^\infty \mathrm{d}y \cdot y^\mu K_{\lambda_0}(ay) = 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\lambda_0}{2}\right)\Gamma\left(\frac{1+\mu-\lambda_0}{2}\right).$$

The solution of equation (39) is given by

$$W(x,s) = C(s)y^{\delta_0}K_{\lambda_0}(y) + y^{\delta_0}Z^*(y), \qquad y = A(s)x^v, \tag{53}$$

where

$$Z^*(y) = \mathrm{e}^{-\int A_2\mathrm{d}y}\int \mathrm{e}^{\int(A_2-A_1)\mathrm{d}y}\left(\int f(y)\,\mathrm{e}^{\int A_1\mathrm{d}y}\,\mathrm{d}y\right)\mathrm{d}y, \tag{54}$$

$$f(y) = -\frac{-y^{\delta_0}}{s}\delta\left(\left(\frac{y}{A(s)}\right)^{1/v}\right) = -\frac{y^{-\delta_0}}{s}\delta(x). \tag{55}$$

It is easy to see from equations (6) and (18) (when = 2) that

$$\alpha = \frac{(2+\theta+\theta')\gamma}{2+\theta}, \qquad B = \frac{A_\alpha\Gamma(1-\alpha)}{v^2}\left(\frac{2C_{\lambda_0}}{C'_{\lambda_0}}\right)^v\left(K_\gamma^\theta\right)^{v-1}, \tag{56}$$

since $\int_0^\infty \mathrm{d}x \cdot x^{d_f-1}x^2W(x,s) = 2K_\gamma^\theta\big/s^{2\gamma/(2+\theta)+1}$ corresponding to the Laplace transform of $\langle(\Delta x)^2\rangle(t)$, $C'_{\lambda_0} = \int_0^\infty \mathrm{d}y \cdot y^{\frac{d_f+2}{v}+\delta_0-1}K_{\lambda_0}(y)$.

Let us now turn to discuss the asymptotic behaviour of $W(x,t)$ as predicted by equation (53). Noting that $Z^*(y) = 0$ when $x > 0$, it follows from (53) and (49) that

$$W(x,s) \approx G''s^{-(1-\alpha d_f/2v)}\frac{1}{(xs^{\alpha/2v})^\kappa}\exp\{-(xs^{\alpha/2v}/G''')^v\} \tag{57}$$

for $|xs^{\alpha/2v}| \gg 1$, where $G'' = \sqrt{\frac{\pi}{2}}v^{1-(d_f+\kappa)/v}\big/C_\lambda\big(GK_\gamma^\theta\big)^{(d_f+\kappa)/2v} > 0$, $G''' = \big(v\sqrt{GK_\gamma^\theta}\big)^{1/v}$, $\kappa = v\big(\frac{1}{2} - \delta_0\big)$.

By the same method of [31, 32], we expected that

$$W(x,t) \sim t^{-(\alpha d_f/2v)}(x/X_\theta)^{\delta'} \exp\{-\text{const} \times (x/X_\theta)^{u'}\}, \tag{58}$$

when $\xi \equiv x/X_\theta \sim x/t^{\gamma/(2+\theta)} \gg 1$ and $t \to +\infty$. The Laplace transform of (58) can be evaluated by applying the method of steepest descent, and the result is compared with (57). This yields

$$u' = v \Big/ \left(1 - \frac{\alpha}{2}\right) \tag{59}$$

and

$$\delta' = u' \left[\tfrac{1}{2}(\alpha d_f/v - 1) - \kappa\right]. \tag{60}$$

It is interesting that if $\theta = \theta' = 0$, $\nu = 0$, we have $u' = 1\big/\left(1 - \frac{\gamma}{2}\right)$ and $\delta' = u'\left[\frac{1}{2}(\gamma d_f - 1)\right]$, which are same as that of FFPE in [33, 34] under the condition $W(x, 0) = 0$.

This shows that if $d_f - 1 - 2v > 0$, i.e. $\nu + \mu > -(d_f + 1 - \theta')$, the stretched Gaussian universality holds for the fractional Fokker–Planck equations (14) as expected by R Metzler and J Klafter.

Remark. From the above discussion, we can see that the results (58)–(60) for the asymptotic behaviour of $W(x, t)$ also hold under the initial condition $W(x, 0) = 0$.

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